# GROUPS GENERATED BY TWO OPERATORS EACH OF WHICH TRANSFORMS THE SQUARE OF THE OTHER INTO A POWER OF ITSELF.

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Two special cases of the category of groups defined in the heading of this paper have been considered, viz., when the square of each of the two generating operators ($s_1$, $s_2$) is transformed either into itself or into its inverse by the other.[1] In each of these cases it was found that the orders of $s_1$, $s_2$ do not have an upper limit. It will be found that both of these orders must always have an upper limit unless at least one of these operators transforms the square of the other either into itself or into its inverse.

The given conditions give rise to the following equations:

$$s_1^{-1}s_2^2s_1 = s_2^a, \quad s_2^{-1}s_1^2s_2 = s_1^b.$$

If at least one of the two numbers $a$, $b$ were odd the order of the corresponding operator would be odd, and hence the group ($G$) generated by $s_1$, $s_2$ could be generated by a cyclic group of odd order and an operator transforming this group into itself. As many of the properties of such a group are known we shall confine our attention in what follows to the cases when both $a$ and $b$ are even numbers, and hence we shall assume that the conditions under consideration are written in the form

$$s_1^{-1}s_2^2s_1 = s_2^{2\alpha}, \quad s_2^{-1}s_1^2s_2 = s_1^{2\beta}.$$

Some fundamental properties of $s_1$, $s_2$ may be deduced from these equations in the following manner:

$$s_1^{-2}s_2^2s_1^2 = s_2^{2\alpha^2}, \quad s_2^{-2}s_1^2s_2^2 = s_1^{2\beta^2}, \quad s_2^{-2}s_1^{-2}s_2^2 = s_2^{2(\alpha^2-1)}s_1^{-2},$$

$$s_1^{-2\beta^2} = s_2^{2(\alpha^2-1)}s_1^{-2}, \quad s_1^{2(\beta^2-1)}s_2^{2(\alpha^2-1)} = 1.$$

[1] Cf. *Paris Comptes Rendus*, Vol. 149 (1909), p. 843.

From the last equation it follows that each of the two operators $s_1^{2(\beta^2-1)}$, $s_2^{2(\alpha^2-1)}$ is invariant under $G$ and therefore it results from the first set of equations that

$$s_1^{2(\beta^2-1)} = s_1^{2\beta(\beta^2-1)}, \quad \text{or} \quad s_1^{2(\beta-1)(\beta^2-1)} = 1 = s_2^{2(\alpha-1)(\alpha^2-1)}.$$

By combining the last equation with

$$s_1^{2(\beta^2-1)} = s_2^{2(1-\alpha^2)}$$

it results that

$$s_1^{2(\alpha-1)(\beta^2-1)} = 1 = s_2^{2(\beta-1)(\alpha^2-1)}.$$

By transforming $s_2^2$ by $s_1^{2(\beta^2-1)}$ and $s_1^2$ by $s_2^{2(\alpha^2-1)}$ it is clear that the orders of $s_1$, $s_2$ must divide respectively

$$2(\beta^{2(\alpha^2-1)} - 1) \text{ and } 2(\alpha^{2(\beta^2-1)} - 1).$$

The orders of $s_1$, $s_2$ must therefore divide the highest comon factor of

$$2(\beta - 1)(\beta^2 - 1), \quad 2(\alpha - 1)(\beta^2 - 1), \quad 2(\beta^{2(\alpha^2-1)} - 1)$$

and

$$2(\alpha - 1)(\alpha^2 - 1), \quad 2(\beta - 1)(\alpha^2 - 1), \quad 2(\alpha^{2(\beta^2-1)} - 1)$$

respectively.

The subgroup ($H$) generated by $s_1^2$, $s_2^2$ is evidently invariant under $G$ and the corresponding quotient group is dihedral. As the commutator subgroup of $H$ is composed of invariant operators under $G$ the fourth derived of $G$ is always the identity.

§ 2. *Groups generated by two operators which satisfy one of the following sets of conditions:*

$$s^{-1}s_2^2s_1 = s_2^4, \quad s_2^{-1}s_1^2s_2 = s_1^4; \qquad s_1^{-1}s_2^2s_1 = s_2^{-4}, \quad s_2^{-1}s_1^2s_2 = s_1^{-4}.$$

The first set of relations implies that $\alpha = \beta = 2$. Hence it results from the preceding section that the orders of $s_1$, $s_2$ must divide 6. If these orders are 6 $H$ is of order 9 and $s_1s_2$ transforms the operators of $H$ into their inverses. Hence $s_1s_2$ is of even order. If this order is 2 the group generated by $H$ and $s_1s_2$ is of order 18, and it is completely determined by the facts that it contains the non-cyclic group of order 9 and an operator which transforms each operator of this non-cyclic group into its inverse. In this case $G$

is of order 36 and such a $G$ can clearly be generated by the two substitutions

$$s_1 = abc \cdot de, \quad s_2 = ab \cdot def.$$

As we may annex to these substitutions any constituents of order 2 in new letters it results that the order of $s_1s_2$ is an arbitrary even number and hence the order of $G$ is an arbitrary multiple of 36. To prove that there is only one such group of a given order the following considerations are helpful. The cyclic group generated by $(s_1s_2)^2$ has only the identity in comomn with $H$ since $s_1s_2$ transforms each operator of $H$ into its inverse. This cyclic group is invariant under $G$. In fact each of its generators is transformed into itself by half the operators of $G$ and into its inverse by the rest of these operators since $(s_2s_1)^2 = (s_1s_2)^{-2} = s_2^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = s_2^2s_2^{-2} \cdot s_2^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = s_2s_1^{-1}s_2s_1^{-1}$. It is now clear from the general theory[2] of simply isomorphic groups that if $s_1$, $s_2$; $s_1^1$, $s_2^1$ are two pairs of operators satisfying the given conditions and if they generate groups of the same order these groups are necessarily simply isomorphic. Hence the theorem: *If each of two operators of order 6 transforms the square of the other into its fourth power these operators may be so selected that they generate a group whose order is an arbitrary multiple of 36 and there is only one group of each such order which can be generated by two of its operators of order 6 satisfying the given conditions. The second derived of each of these groups is unity.*

When the order of only one of the two operators $s_1$, $s_2$ is 6 that of the other must be 2, since it transforms the square of the former into its inverse. In this case $H$ is the cyclic group of order 3 and the order of $G$ is an arbitrary multiple of 12. The second derived of all of these groups is the identity since $(s_1s_2)^2$ and $s_1^2$ generate an invariant abelian group and the corresponding quotient group is the four-group. When the order of $s_1$ is 3 that of $s_2$ must be 2 and $G$ is the symmetric group of order 6. If $s_1$, $s_2$ are both of order 2 they may be so selected as to generate an arbitrary dihedral group. Combining these results we have that when $s_1$, $s_2$ are both of order 2 they may generate one and only one group of

[2] *Bulletin of the American Mathematical Society*, Vol. 3 (1897), p. 218.

a given order and this order is an arbitrary even number greater than 2; when one of these operators is of order 6 while the other is of order 2 they may generate one and only one group whose order is a given multiple of 12; when both of them are of order 6 the order of the group generated by them is a multiple of 36 and they may be so selected as to generate a group whose order is an arbitrary multiple of 36, but each such group is completely determined by its order.

If we consider all the possible groups which can be generated by two operators satisfying the two conditions under consideration it results from the above that there are exactly three such for every order which is a multiple of 36, one of these is dihedral, another is generated by an operator of order 6 and an operator of order 2, while the third is generated by two operators of order 6. When the order is divisible by 12 but not by 36 there are two and only two such groups, while the dihedral group is the only one that can be generated by two such operators whenever the order is any other even number greater than 2.

When $s_1$, $s_2$ satisfy the relations $s_1^{-1}s_2^2s_1 = s_2^{-4}$, $s_2^{-1}s_1^2s_2 = s_1^{-4}$ their orders must divide 18 according to the general formulas of the preceding section. The following substitutions prove that these orders may be exactly 18:

$$s_1 = acegibdfh \cdot jk, \quad s_2 = abidecghf \cdot lm.$$

As $s_1s_2 = agd \cdot bch \cdot jk \cdot lm$ the order of the group generated by these two substitutions is 108. The smallest group that can be generated by two operators of order 18 which satisfy the given conditions is of order 54 and such a group is clearly generated by the two substitutions $acegibdfh \cdot jk$, $abidecghf \cdot jk$.

In general, $s_1^3$ is commutative with $s_2^2$ and hence with every operator of $H$. Similarly, it may be observed that $s_2^3$ is commutative with every operator of $H$. From this it results that every operator of the dihedral group generated by $s_1^9$, $s_2^9$ is commutative with every operator of $H$. It is also clear that these groups can have only the identity in common since the former contains no invariant operator of odd order. It therefore results that $G$ must involve the direct product of $H$ and this dihedral group. As these two groups clearly generate $G$ it results that $G$ is this direct product. That is, *if two operators of order* 18 *satisfy the conditions* $s_1^{-1}s_2^2s_1 = s_2^{-4}$, $s_2^{-1}s_1^2s = s_1^{-4}$ *they generate the direct product of a dihedral group and a certain non-abelian group of order 27. Moreover, every such direct product can be generated by two operators of order* 18 *satisfying the given conditions.*

If only one of the two operators $s_1$, $s_2$ is of order 18 the other must be of order 9 since $s_1^2$, $s_2^2$ cannot be commutative. That is, if an operator of order 18 and an operator whose order is not 18 satisfy the conditions $s_1^{-1}s_2^2s_1 = s_2^{-4}$, $s_2^{-1}s_1^2s_2 = s_1^{-4}$ these operators must generate the direct product of the group of order 2 and a certain group of order 27. When $s_1$, $s_2$ are both of order 9 they generate a group of order 27, and when both are of order 2 they generate a dihedral group. Combining these results we have that if two non-commutative operators satisfy the two conditions $s_1^{-1}s_2^2s_1 = s_2^{-4}$, $s_2^{-1}s_1^2s_2 = s_1^{-4}$ their orders have one of the following pairs of values 18, 18; 18, 9; 9, 9; 2, 2. In the first and last of these cases $G$ may be any one of an infinite system of groups; viz., any dihedral group in the last case, and the direct product of such a group and a certain group of order 27 in the first case. In each of the other two cases there can be only one group, viz., the non-abelian group of order 27 which involves operators of order 9 in the third case, and the direct product of this group and the group of order 2 in the second case.

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